TAIL-RISK AND THE ST PETERSBURG COIN-FLIP MODEL

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ABSTRACT

Though a St. Petersburg-like gamble would not be entered into by the party to either side of such gamble in real life, in this paper we use a plausible version of this long-tailed game to illustrate how entities could earn positive expected returns for both themselves and their clients. The paper applies this simple gamble to a depository institution. A bank receives default premiums from borrowers, but must make frequent small payoffs, somewhat rare large payoffs, and also extremely rare enormous ones when loans default. Both banks and their borrowers can have positive expected gains, if the banks can shift the cost of extremely rare losses onto their insurers. A simulation based upon the St. Petersburg coin-flip model shows that self-interested bank owners with limited liability seek loans with tail-risk.

INTRODUCTION

A coin-flip model, based on the process underlying the famed St. Petersburg Paradox gamble, [10] can build intuition about financial instruments with long-tailed (also called fat-tailed or heavy-tailed) outcomes [9]. Limited liability firms as well as units within such firms could find long-tailed outcomes useful to capture gains while transferring infrequent, but huge, losses to others. When information about the extreme tails is hidden, imperfect or ignored, extreme losses may appear to be an unpredictable accident. Also, if these losses impose unacceptable damage on other entities, outside support may be elicited.

The gamble of the classic St. Petersburg Paradox would never be credibly offered by any seller and no buyer would pay the expected value to play it [12]. Sellers cannot credibly offer the impossibly huge amounts required in extreme outcomes. Buyers would willingly pay only a small fraction of the expected value. However, a truncated version, one with limits to the maximum payouts, could attract both buyers and sellers if the gambles were subsidized by shifting rare, but huge payouts to others. Sellers could profitably charge premiums below the expected value of the payouts to be paid to the buyers, but above the expected losses the sellers would bear themselves. The subsidy can occur in strictly private markets with imperfect information as well as in markets subject to government intervention. In private markets, because of information asymmetry or disaster myopia, depositors and insurers may under price tail-risk. Politically well-connected firms are more likely to be bailed out than their non-connected peers [5].

BANK LOANS WITH LONG-TAILED RISKS

A bank that invests in risky loans or securities with long-tailed outcomes and that finances these assets with a combination of equity capital from the owners and debt in the form of deposits provides one example that could be modeled by a long-tailed gamble. The bank's depositors, due to either informational asymmetries or to disaster myopia or both, ignore the rare risk of insolvency of the bank. They may treat the very low probability that losses on the banks assets exceed its equity capital as if it were zero [13] [15]. The depositors guarantee the losses, since their deposits cover all losses exceeding the owners' equity capital. Since the bank's depositors know less about the tail-risk than the bank's risk management, they can not determine a correct risk premium and may demand too high a risk premium for less risky banks – the "lemons problem" [2] [4]. In this model, banks face solvency, but not liquidity, risk. Depositors do not know the results until the end of the each period's coin toss and cannot act in anticipation to create runs.

Since higher risk investments yield higher expected returns, raising low risk money and investing it at higher risk yields positive expected spreads. Creating profitable spreads from investments with long-tailed negative payoffs distributions takes advantage of imperfect information about very small probability outcomes [1]. For example, suppose a unit of a bank creates an investing opportunity with an attractive spread that is in fact based upon tail-risks. The bank's risk management might underestimate the expected losses of the long-tailed risks, due to insufficient data, biased models, lack of internal authority, or management disinterest in extremely low probability outcomes, and charges the unit an insufficient risk premium. The unit "earns" a spread and accompanying generous compensation. Creating tail-risk can be advantageous to the bank's owners.

MODEL

A truncated St. Petersburg paradox gamble models the above example as well as other examples discussed in a longer version of this paper. The size of the equity capital defines the upper limit of the gamble for the bank. A guarantor covers the portion of any payouts exceeding the company's capital up to a larger limit; this defines the upper limit for the customer. For simplicity, the firm maintains a constant amount of equity capital. At the end of each period, the bank makes all payouts, in the form of defaults, required and collects the premiums for the next gamble. Whenever the premium collected exceeds the amount of the defaults, the firm distributes the difference as a dividend. Whenever the payouts exceed the premiums, it raises the difference as additional equity. Whenever the losses exceed its equity, the firm fails and the guarantor covers all remaining losses. For simplicity, there is one guarantor who could be uninsured depositors, an explicit outside insurer, or an implicit one [7] [3]. The size of firm's equity limits its owners' liability, who can raise new capital, form a new firm, and resume.

The customers receive a payout $Q = X \cdot 2^N$, where N equals the number of consecutive coin tosses until the first head occurs. The stakes parameter 'X' is chosen to equal \$1 million. The limit for the owner is set at N*=10 tosses for a maximum Q*equal to $\$2^{10}$ million (\$1,024 million). The limit for the guarantor equals N**=15 tosses for a maximum of $\$2^{15}$ million (\$32,768 million). The guarantor's maximum payment equal Q**-Q* ($\$2^{N^{**}}$-$\$2^{N^*}$million). A period of one month is chose for a gamble with a fair coin that is tossed until a head occurs. If the first head occurs on the 10^{th} toss, the firm makes its maximum payout of $\$2^{10}$ million and recapitalizes, but the guarantor suffers no losses. If the first head occurs on the 11^{th} or higher toss, the firm also pays $\$2^{10}$ million, the firm makes the payout of $\$2^{10}$ million and recapitalizes, but the guarantor assumes all payouts exceeding the $\$2^{10}$ million up to its limit of $\$2^{15}$ million. The guarantor's maximum payout of $31,744 million, ($\$2^{15}$-$\$2^{10}$) million, occurs whenever the first head does not occur on the first 14 tosses, regardless of the outcome of the 15^{th} toss. The owner's capital ratio, total equity to maximum payout, equals 3.125% ($\$2^{10}$ million/$\$2^{15}$ million). The chance of 10 heads occurring, N*=10, is 2^{-10} or 0.098%. The reciprocal, the expected time to occurrence, equals 1,024 months or 85 1/3 years.

The expected value of the unbounded version of the game is undefined:

$$E(V)=\sum_{N=1}^{\infty}\frac{1}{2^N}2^N=\infty. \quad (1)$$

However, since the game is bounded by a maximum payout equal Q** = $\$2^{N^{**}}$ million, then, for N**>1, the expected value of the payout for the customers, [E(V**)], is:

$$E(V^{**})=\sum_{N=1}^{N^{**}}\frac{1}{2^N}2^N=\left[\sum_{N=1}^{N-1^{**}}\frac{1}{2^N}2^N\right]+\frac{1}{2^{N^{**}}}2^{N^{**}}+\frac{1}{2^{N^{**}}}2^{N^{**}}.$$
$$E(V^{**})=N^{**}-1+1+1=N^{**}+1. \quad (2)$$

$$E(V^*)=\left[\sum_{N=1}^{N^*-1}\frac{1}{2^N}2^N\right]+\frac{1}{2^{N^*}}2^{N^*}+\frac{1}{2^{N^*}}2^{N^*}.$$
$$E(V^*)=N^*-1+1+1=N^*+1. \quad (3)$$

Using above equations, the expected value of the total payout for each gamble, [E(V**)], equals $16 million[Eq. 3], but the expected value borne by the bank's owners, [E(V*)], equals $11 million [Eq. 3]. While the payoff covered by the guarantor might seem unforeseeable by its rarity, the expected value of the covered losses is N**-N*- Φ or $(5 – Φ) million each month in this example, where Φ is the guarantor's premium. If the guarantor charges a default premium [Φ] equal to $5 million or greater than $5 million per month, there would be neither subsidies nor expected gains. In the simulation, the Φ is assumed to be zero, for simplicity. As long as bank depositors receive a risk premium that is less than sufficient compensation for their tail-risk, they subsidize lending [3]. For illustrative purposes the monthly default premium paid by the borrowers to play, is set in the middle at $13.50 million of the expected values of the game in this example, half way between the $16 (N**+1) and $11 (N*+1). Simulations based on other default premiums result in similar outcomes for the guarantor.

SIMULATION

Table 1 reports 10 simulations of 40 independent monthly lotteries over a 100-year period. Each simulation reports the results of a total of 48,000 lotteries (12 x 100 x 40). The simulations show that both selling firms and customers can generally expect positive earnings as a group over a long period. The variability of outcomes is much less volatile for the firms than it is for the customers. Out of 10 simulations the profit for owners in all simulations is positive. The profit for the customers is positive in 9 out of 10. In all 10 simulations, the guarantor has losses. While both parties are likely to earn positive returns, the customers have higher variability and some risk of loss, even over a long time period. The average loss to the guarantor equals $248,934 million for the century. The expected value is $250,000 million ($5 million x 48 thousand lotteries). These losses are the gains to be divided between the firm's owners and their customers. Their relative share depends upon the default premium.

Table 1

Owners' Earnings	Customers' Defaults	Guarantor's Losses
$ 120,454	$ 128,378	$248,832
$ 132,622	$ 155,122	$287,744
$ 100,230	$ 147,578	$247,808
$ 108,924	$ 149,124	$258,048
$ 136,396	$ (22,732)	$113,664
$ 124,448	$ 20,960	$145,408
$ 119,440	$ 261,488	$380,928
$ 129,188	$ 122,716	$251,904
$ 107,926	$ 179,818	$287,744
$ 115,002	$ 152,262	$267,264
	Average	
$ 119,463	$ 129,471	$ 248,934

DISCUSSION

In this model, depositors are either insured or are unaware of the tail-risk considered here for all firms. For any given asset-to-equity ratio, $2^{N^{**}}/2^{N^{*}}$, size, per se, does not affect the expected losses, since expected losses must equal N**-N*. If some equity capital for the "too-big-to-fail" firm is not at risk, the effective subsidy increases. A certain amount of equity remains is also guaranteed, conferring competitive advantage over their rivals lacking that status. The "too-big-to-fail" firms can undercut their competitors and acquire long-tailed risks, since they operate with an effectively higher guarantee and can pass on more of the tail-risk to insurers or governments. In this model, being "too-big-to-fail", is similar to having a lower asset-to-equity ratio.

This simple coin-flip model shows how banks could earn positive expected returns for both themselves, under charge their risky borrowers, and shift very rare losses to external parties. Combining limited liability and hidden tail-risk allows owners to gain from less informed depositors, insurers, or from the larger society. A limited liability company has no incentive to reduce the magnitude of losses exceeding its equity capital. With imperfect information, the firm's creditors, depositors, insurers, or government will bear the risk.

To model certain activities as a zero-sum game with long-tailed distribution does not assert that these examples exist. Modeling does, however, illustrate that self-interested parties can reap rewards from self-created tail-risk. Information asymmetry can be associated with long-tailed risks that can be shifted to less informed parties, or to the larger society. Other long-tailed distributions, such as the Pareto and Zipf distributions, could illustrate the points made here as well. The St. Petersburg coin-flip model is useful for its prominent presence in a wide variety of research areas, familiarity and intelligibility to a wide audience.

The model cannot show that actual speculative banking behavior creates and exploit long-tailed investments to gain. However, other observers believe is valid.

> "If they [bankers] look conservative, it's only because their loans go bust on rare, very rare occasions. There is no way to gauge the effectiveness of their lending activity by observing it over a day, a week, a month, or … even a century! . . . They are not conservative at all; just phenomenally skilled at self-deception by burying the possibility of a large, devastating loss under the rug." [14, p. 296].

> "Firms like AIG, Bear Stearns, Citigroup, and Lehman Brothers took risks that were virtually unbounded, albeit low in probability. The most obvious factor driving this behavior seems to be the compensation system, which typically paid hefty bonuses when employees made gains but did not penalize them significantly when they incurred losses. The profitable one-sided bet this offered employees was known variously as the Acapulco Play, IBG (I'll be gone if it doesn't work), and, in Chicago, the O'Hare Option (buy a ticket departing from O'Hare International Airport: if the strategy fails, use it; if the strategy succeeds, tear up the ticket and return to the office). That such strategies were common enough in the industry as to have names suggests that not all traders were oblivious to the risks they were taking." [11, p. 139]

It does show that income gained in self-interested endeavors tail-risk seeking behavior by limited liability firms can be a negative externality. Alan Greenspan famously testified to an error in his thinking, stating, "I made a mistake in presuming that the self-interests of organizations, specifically banks and others, were such as that they were best capable of protecting their own shareholders and their equity in the firms." [6]. Self-interested banks in this model are less interested in protecting their equity than is an outside regulator.

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