# RELATIONSHIP BETWEEN RETURN FRACTALITY AND BIPOWER VARIATION: A COMMENT

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## ABSTRACT

Realized volatility of stock returns results from a smooth stochastic process and discontinuous jumps. The jump component is often measured by subtracting bipower variation of high frequency data from realized variation. However, measurement of the bipower variation depends on trading volume. Generally, when the stock return follows a Wiener process, so does the trading volume. It turns out the relative trading volume affects the size of return fractality, and thus, stock returns appear mostly characterized by jump elements. To the extent that the stock's return fractality is measured by its fractal dimension, the paper also offers an alternative approach to computing fractal dimension.

## INTRODUCTION

The standard Wiener process as is often purported to be the manifestation of random walk assumes stochastic continuity, and thus, seems to miss some important elements in describing stock price behaviors, when stock price process is characterized by occasional discontinuous jumps, e.g. Merton [11]. Efforts have been made to explain how occasional jumps can converge to a long-run level of volatility rate, see for example, Bollerslev [5], Nelson [6] and Engle, *et al.* [7].

Recently, there have been several studies to measure the jump component of stock price processes, e.g. Barndorff-Nelson [3,4] and Andersen, *et al.* [2]. In these studies, the stock's realized volatility follows from the quadratic variation process as the increment as the sampling frequency of the underlying returns increases. It is also shown that the realized volatility can be decomposed asymptotically into the *bipower variation* attributable to smooth processes and another variation, which arises from the discontinuous jump process. Thus, one can measure the jump variation by subtracting bipower variation from realized volatility. Clearly, depending upon how the bipower variation is computed, the computed jump size can be widely different from what it really is. In this paper, we show that the major part of the realized volatility may come from the discontinuous jumps rather than smooth stochastic bipower variation, if we recognize the existence of the return fractals.

## ECONOMICS OF RETURN FRACTALITY

We often multiply the return over a short time step, , by time intervals to cover the entire longer term horizon, , to compute the return over , , i.e.

The rationale for doing this is that whatever happens over a time period can be extrapolated by whatever happens during linearly by , as . The daily, monthly or quarterly returns are annualized in this way, but this simplification has no predictive content.

Realistically, the factor that is multiplied by, i.e. , cannot be used to forecast the return over , i.e. . Perhaps, we should not use the simple calendar time period to extrapolate a shorter-term return to get a longer-term return. The appropriate time units as applied to the financial market cannot be one-dimensional. Some other "effective" time units, , may be feasible.

To detail our discussion, we remind readers of the fact that with the calendar time dimension being one-dimensional, i.e. the time segment is exactly decomposed into nonoverlapping segments. The length of each segment is represented by a formula . In this case, if an event in each segment can be deduced from the whole linearly, each segment is "similar" by a ratio .

What if each calendar time period has different characteristics influenced by many economic factors so that each day is not the same day? In other words, the time unit that we deal with is a two-dimensional "object," for example, , which can be describable by a two-tuple pair of some unknown variables and such that . Thus, if each segment is represented by and , where then it is given by a formula, . That is, the size of each decomposed timepiece can be described just as we dimension a rectangular object with and each of these "rectangular" parts is deducible from the whole by a similarity of ratio

Generalizing this pattern of similarity, a -dimensional time unit as applied to the stock return can then be decomposed into subpieces from the whole by a similarity of ratio See Mandelbrot [9]. In mathematics, the exponent, , is known as the Hausdorff dimension named after a German mathematician, Felix Hausdorff or Hurst exponent, [8]. Interestingly, however, the Hausdorff dimension can be fractional, and hence, named "fractal dimension." Furthermore, the more sophisticated an object is, the higher the value of the fractal dimension.

To explore the concept further, imagine that a long time period is divided into time intervals for a similarity ratio . Solving the expression for ,

Eqn (2) states that the value of in eqn (1) depends on the quantity, and the fractal dimension . Generally, nor , and hence, . We define the expression in the parenthesis as

being the reciprocal of the similarity ratio then represents the total number of times that a given return over a short time period can be replicated to produce an annual return. That is, the value provides information as to how for example, a year or even a day should be divided, or the appropriate total number of "economic" or "effective" time periods in a year or in a day.

Assuming that represents a year or a day, i.e. , for convenience and hence the annual (or daily) return , our return forecasting model is

Typically, this concept of the fractal dimension in the financial market does not necessarily result in the squared variation, when the fractal dimension is not one (1). We now relate eqn (4) to the theory of bipower variation as follows.

## BIPOWER VARIATION

Consider the following quadratic variation for the cumulative return process:

Define, for example, the daily-realized volatility or variation by the summation of the corresponding high frequency intra-daily squared returns, where is the number of periods and is assumed to be an integer, i.e.

Then, the realized volatility is the increment of quadratic variation process as the sampling frequency of the underlying returns increases. That is,

Barndorff-Nielson and Shephard [3] introduces a general form of variation, where the absolute values of two subsequent intraday high-frequency or daily security returns are multiplied together to produce what they term the realized bipower variation, i.e.

It is shown that in the case, this will, up to a simple known multiple, converge to the same probability limit as realized variance when prices follow a stochastic volatility process and that for this bipower variation, the limit does not change with the addition of rare jumps. Thus, substituting eqn (4) into eqn (8), the standardized realized bi-power variation can now be defined as:

Then, as the sampling frequency increases, the realized bipower variation approaches to:

Combining the results in eqn (7) and eqn (10), we conclude that

With non-negativity in jumps,

The conclusion is that the higher the fractal dimension, the lower the bipower variation and hence, the greater the jump for any given realized volatility. Thus, we would expect much larger jumps in magnitude than what we may see from computing the bipower variation in a traditional sense, if the stock carries the fractal dimension exceeding one (1).

## THE IMPLIED STOCK PRICE PROPERTIES

Assuming that the daily trading horizon is 390 minutes, the log price relatives can be represented heuristically as

The symbol is the continuously compounded rate of return from to . Assuming that all one-period returns are of stochastically independent identical distribution such that and , the central limit theorem states that

Define . Then,

Now suppose that the stock price follows some modified Wiener process with a daily (or an annual) drift and a daily (or an annual) volatility , where and . Then

where with and represents a normal distribution. As usual, The log price in the Itô process is

Assuming that , the expected value and the variance of are and respectively.

## FRACTAL DIMENSION AND VOLATILITY

In the past, the issue of volatility has been dealt with along the lines of either GARCH models or fractal returns. Despite some known methods of approximating the fractal dimension, e.g. Mandelbrot, *et al*. [10] and Wavelet transforms based on a Taylor series for a time series signal , we take somewhat simpler approach to measuring fractal dimensions. We conjecture that the fractal feature of securities returns, which determines the "effective" time period for forecasting, can be explained by the volume of trade. In other words, we regard the effective time period as the "trading time." We now define more formally securities return fractal as follows.

Suppose that in eqn (4). Then, for any , a stock return has a dimension , if and only if there is some positive constant such that

Eqn (15) is equivalent to solving for by taking the limit as

Our analysis is then that given a fractal dimension , the trading volume, which determines the value depends on the time step . To model it, assume that the relative trading volume is and

Eqn (17) states that the higher the value of , and the smaller the time step , we expect the lower relative trading volume. If , clearly , which is constant. A regression equivalent to eqn (17) is

Alternatively, eqn (17) also states that the fractal dimension is a function of both and That is,

If is a Wiener process, the security's fractal dimension also follows the Itô process, and it is easy to show the expected mean and the volatility of a change in .

## SUMMARY AND CONCLUSION

Realized volatility of stock returns results from a smooth stochastic process and discontinuous jumps. The jump magnitude is measured by subtracting bipower variation from realized variation. However, the presence of the return fractality reduces the bipower variation raising the importance of the jump component of realized volatility. In addition, it is possible that the fractal dimension itself can follow its own stochastic process.

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